

An extension of Paulsen-Gjessing's risk model with stochastic return on investments

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Abstract We consider in this paper a general two-sided jump-diffusion risk model that allows for risky investments as well as for correlation between the two Brownian motions driving insurance risk and investment return. We first introduce the model and then find the integro-differential equations satisfied by the Gerber-Shiu functions as well as the expected discounted penalty functions at ruin caused by a claim or by oscillation; We also study the dividend problem for the threshold and barrier strategies, the moments and moment-generating function of the total discounted dividends until ruin are discussed. Some examples are given for special cases.

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1 Introduction

The study of insurance risk models with stochastic return on investments has attracted a fair amount of attention in recent years, for example, Paulsen (1993) proposed the following general risk process U_t that allows for a stochastic rate of return on investments as well as a stochastic rate of inflation:

$$U_t = \frac{\mathcal{E}(R)_t}{\mathcal{E}(I)_t} \left(u + \int_0^t \frac{\mathcal{E}(I)_{t-}}{\mathcal{E}(R)_{t-}} dP_s \right).$$

The notation $\mathcal{E}(A)$ denotes the Doléans-Dade exponential of A given as the solution of the stochastic differential equation $d\mathcal{E}(A)_t = \mathcal{E}(A)_{t-} dA_t$ with $\mathcal{E}(A)_0 = 1$, and P_t, I_t and R_t are all semimartingales representing the surplus generating process, the inflation generating process and the return on investment generating process, respectively. The initial values are $P_0 = u, I_0 = 0$ and $R_0 = 0$. He obtained an integro-differential equation and an analytical expression for ruin probability under certain conditions. Paulsen and Gjessing (1997a) simplified the model above by assuming that there is no inflation and both the surplus P_t and the return on investment R_t are independent classical risk processes perturbed by Brownian motions. Paulsen (1998a) considered a risk process U_t given by

$$U_t = u + P_t + \int_0^t U_{s-} dR_s, \text{ with } P_0 = R_0 = 0, \quad (1.1)$$

where P_t and R_t are independent Lévy processes. With the above notation, the solution of (1.1) can be written as $U_t = \mathcal{E}(R)_t(u + \int_0^t \mathcal{E}(R)_{s-}^{-1} dP_s)$. Cai and Xu (2006) considered a risk model that assumed the surplus of an insurer follows a jump-diffusion process and the insurer would invest its surplus in a risky asset, whose prices are modeled by a geometric Brownian motion. In the Discussion of the paper, Hailiang Yang extended the model of Cai and Xu to the case in which the surplus can be invested in both risky and risk-free assets.

For some related discussions, among others, we refer the reader to Cai (2004), Yuen, Wang and Ng (2004), Cai and Yang (2005), Yuen and Wang (2005), Zhang and Yang (2005), Yuen, Wang and Wu (2006), Meng, Zhang and Wu (2007). For further references see two survey papers Paulsen (1998b) and Paulsen (2008). Some recent papers extended the model to renewal risk models with stochastic return, see e.g. Gao and Yin (2008) and Li (2012).

Motivated by the previously mentioned papers, the aim of present paper is to generalize the model given in (1.1) by considering that P_t and R_t are general two-sided jump-diffusion

risk models that allow for risky investments as well as for correlation between the two Brownian motions driving insurance risk and investment return. The rest of the paper is organized as follows. In Section 2 we introduce the model. Integro-differential equations for the Gerber-Shiu functions are established in Section 3 and, in Section 4, we study the dividend payments under the threshold and barrier strategies. Finally, we give the concluding remarks.

2 THE MODEL

Assume that the surplus generating process P_t at time t is given by

$$P_t = u + pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_{P,t}} S_{P,i}, \quad t \geq 0, \quad (2.1)$$

where u is the initial surplus, p and σ_P are positive constants, $\{W_{P,t}\}_{t \geq 0}$ is a standard Brownian motion independent of the homogeneous compound Poisson process $\sum_{i=1}^{N_{P,t}} S_{P,i}$, while $\{S_{P,i}\}$ is a sequence of independent and identically distributed random variables. Unlike the model in Paulsen and Gjessing (1997a), we assume that $S_{P,i}$ take values in $(-\infty, +\infty)$. The upward jumps can be explained to be the random gains of the company, while the downward jumps are interpreted as the random loss of the company. Let λ_P be the intensity of Poisson process $N_{P,t}$, and F_P be the common distribution of $S_{P,i}$. We assume throughout this paper that $E[S_{P,i}] < \infty$ and $p - \lambda_P E[S_{P,i}] > 0$.

Now, suppose that the insurer would invest its surplus a risky asset, whose price is assumed to follow the stochastic differential equation $dS(t) = S(t-)dR_t$, where R_t is the return on investment:

$$R_t = rt + \sigma_R W_{R,t} + \sum_{i=1}^{N_{R,t}} S_{R,i}, \quad t \geq 0, \quad (2.2)$$

where $\{W_{R,t}\}_{t \geq 0}$ is another standard Brownian motion, independent of the homogeneous compound Poisson process $\sum_{i=1}^{N_{R,t}} S_{R,i}$, while r and σ_R are positive constants. The intensity of $N_{R,t}$ is denoted by λ_R , and the distribution function of the jump S_R by F_R . Unlike the model in Paulsen and Gjessing (1997a), $W_{P,t}$ is correlated with $W_{R,t}$ and $W_{R,t}$ can be written as $W_{R,t} = \rho W_{P,t} + \sqrt{1 - \rho^2} W_{P,t}^0$, where $\rho \in [-1, 1]$ is constant, $W_{P,t}^0$ is a standard Brownian motion independent of $W_{P,t}$. When $\rho^2 = 1$, there would only be one source of randomness in the model.

We define the risk process U_t as the total assets of the company at time t under this investment strategy, then U_t is the solution of the stochastic differential equation

$$dU_t = dP_t + U_{t-}dR_t, \quad t \geq 0. \quad (2.3)$$

By using Theorem 1 in Jaschke (2003) it is not hard to see that the solution of (2.3) is given by

$$U_t = \mathcal{E}_1(R)_t \left(u + \int_0^t \mathcal{E}_1(R)_{s-}^{-1} dP_s - \rho \sigma_P \sigma_R \int_0^t \mathcal{E}_1(R)_{s-}^{-1} ds \right), \quad (2.4)$$

where

$$\mathcal{E}_1(R)_t = \exp \left\{ \left(r - \frac{1}{2} \sigma_R^2 \right) t + \sigma_R W_{R,t} \right\} \prod_{i=1}^{N_{R,t}} (1 + S_{R,i}).$$

Because the quadratic variational processes of

$$\sigma_P W_{P,t} + \int_0^t \sigma_R U_{s-} dW_{R,s} = \int_0^t (\sigma_P + \rho \sigma_R U_{s-}) dW_{P,s} + \int_0^t \sigma_R \sqrt{1 - \rho^2} U_{s-} dW_{P,s}^0$$

and

$$\int_0^t \sqrt{(\sigma_P + \rho \sigma_R U_{s-})^2 + \sigma_R^2 (1 - \rho^2) U_{s-}^2} dB_s$$

are same, where $\{B\}_{t \geq 0}$ is a standard Brownian motion independent of the compound Poisson processes involved, by Ikeda and Watanabe (1981, Theorem 7.2, p.85), they have the same distributions. Thus, in distribution, (2.3) can be written as

$$\begin{aligned} U_t = u &+ \int_0^t (p + r U_s) ds \\ &+ \int_0^t \sqrt{(\sigma_P + \rho \sigma_R U_{s-})^2 + \sigma_R^2 (1 - \rho^2) U_{s-}^2} dB_s \\ &- \sum_{i=1}^{N_{P,t}} S_{P,i} + \int_0^t U_{s-} d \left(\sum_{i=1}^{N_{R,t}} S_{R,i} \right). \end{aligned} \quad (2.5)$$

Using Itô's formula for semimartingale, one finds that the infinitesimal generator \mathcal{L} of $\{U_t\}_{t \geq 0}$ is given by

$$\begin{aligned} \mathcal{L}g(y) &= \frac{1}{2} (\sigma_P^2 + 2\rho\sigma_P\sigma_R y + \sigma_R^2 y^2) g''(y) + (p + ry) g'(y) \\ &+ \lambda_P \int_{-\infty}^{\infty} [g(y - z) - g(y)] dF_P(z) \\ &+ \lambda_R \int_{-1}^{\infty} [g(y + yz) - g(y)] dF_R(z). \end{aligned} \quad (2.6)$$

Remark 2.1. If $F_P(0) = 0$ and $\rho = 0$, then (2.4) reduces (2.4) of Paulsen and Gjessing (1997a) and (2.6) becomes (2.6) of Paulsen and Gjessing (1997a).

3 The Gerber-Shiu functions

In this section, we consider the Gerber-Shiu expected discounted penalty function for the risk process (2.4). The time of ruin of (2.4) is defined as $T = \inf\{t \geq 0 : U_t < 0\}$ with $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$. The ruin probability with an initial surplus $u \geq 0$ is defined as

$$\psi(u) = P(T < \infty | U_0 = u).$$

Note that ruin may be caused by a claim or by oscillation. Denote the ruin probabilities in the two cases by

$$\psi_s(u) = P(U_T < 0, T < \infty | U_0 = u), \quad \psi_d(u) = P(U_T = 0, T < \infty | U_0 = u).$$

Obviously we have

$$\psi(u) = \psi_s(u) + \psi_d(u), \quad u \geq 0.$$

Moreover, when $\sigma_P \neq 0$, it follows from the oscillating nature of the process U_t that

$$\psi(0) = \psi_d(0) = 1 \quad \text{and} \quad \psi_s(0) = 0.$$

Let $w = w(x_1, x_2)$ be a nonnegative bounded measurable function on $[0, \infty) \times [0, \infty)$. As in Gerber-Shiu (1998), the Gerber-Shiu expected discounted penalty function is defined by

$$\phi(u) = E[e^{-\delta T} w(U_{T-}, |U_T|) I(T < \infty) | U_0 = u], \quad (3.1)$$

where $\delta \geq 0$ is a constant and $I(A)$ is the indicator function of event A . Sometimes we write P_u for the probability law of U when $U_0 = u$ and E_u for the expectation with respect to P_u .

Following Wang and Wu (2008), we decompose the Gerber-Shiu function $\phi(u)$ in (3.2) correspondingly into the following two parts:

$$\phi_s(u) = E[e^{-\delta T} w(U_{T-}, |U_T|) I(U_T < 0, T < \infty) | U_0 = u], \quad (3.2)$$

$$\phi_d(u) = w(0, 0) E[e^{-\delta T} I(U_T = 0, T < \infty) | U_0 = u]. \quad (3.3)$$

Obviously, $\psi(u)$, $\psi_s(u)$ and $\psi_d(u)$ are special cases of $\phi(u)$, $\phi_s(u)$ and $\phi_d(u)$, respectively.

For simplicity, we define the operator

$$\begin{aligned} \mathcal{G}h(u) &= \frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2)h''(u) + (p + ru)h'(u) \\ &\quad + \lambda_P \int_{-\infty}^u h(u - z) dF_P(z) + \lambda_R \int_{-1}^{\infty} h(u + uz) dF_R(z). \end{aligned} \quad (3.4)$$

Theorem 3.1. Assume that $\phi(u), \phi_s(u), \phi_d(u)$ are twice continuously differentiable on $[0, \infty)$, where the derivative at $u = 0$ means the right-hand derivative. If $\sigma_P^2 > 0$, $p - \rho\sigma_P\sigma_R - \lambda_P E[S_{P,i}] > 0$, $F_R(-1) = 0$ and $r - \frac{1}{2}\sigma_R^2 > 0$. Then,

(i) $\phi(u)$ satisfies the integro-differential equation

$$(\delta + \lambda_P + \lambda_R)\phi(u) = \mathcal{G}\phi(u) + \lambda_P \int_u^\infty w(u, z - u) dF_P(z), \quad u > 0, \quad (3.5)$$

with boundary conditions

$$\phi(0+) = w(0, 0), \quad \lim_{u \rightarrow \infty} \phi(u) = 0. \quad (3.6)$$

(ii) $\phi_s(u)$ satisfies the integro-differential equation

$$(\delta + \lambda_P + \lambda_R)\phi_s(u) = \mathcal{G}\phi_s(u) + \lambda_P \int_u^\infty w(u, z - u) dF_P(z), \quad u > 0, \quad (3.7)$$

with boundary conditions

$$\phi_s(0+) = 0, \quad \lim_{u \rightarrow \infty} \phi_s(u) = 0. \quad (3.8)$$

(iii) $\phi_d(u)$ satisfies the integro-differential equation

$$(\delta + \lambda_P + \lambda_R)\phi_d(u) = \mathcal{G}\phi_d(u), \quad u > 0, \quad (3.9)$$

with boundary conditions

$$\phi_d(0+) = w(0, 0), \quad \lim_{u \rightarrow \infty} \phi_d(u) = 0. \quad (3.10)$$

Proof. We can prove the theorem following a similar argument as in Yin and Wang (2010) by using Itô's formula. In the following proof, however, we use a more intuitive infinitesimal argument as in Cai and Yang (2005) and Cai and Xu (2006), where the ruin probabilities have been studied. The main difference from theirs is that our model has two Poisson processes and has two-sided jumps.

(i) Let

$$Y_t = ue^{\Delta_t} + pe^{\Delta_t} \int_0^t e^{-\Delta_s} ds + \sigma_P e^{\Delta_t} \int_0^t e^{-\Delta_s} dW_{P,s}, \quad (3.11)$$

where

$$\Delta_t = \left(r - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t}.$$

Consider the risk process U_t , defined by (2.4), in an infinitesimal time interval $(0, t]$. Since both $N_{P,t}$ and $N_{R,t}$ are Poisson processes, there are five possible cases.

(i). Both $N_{P,t}$ and $N_{R,t}$ have no jumps in $(0, t]$ (the probability that this case occurs is $e^{-\lambda_P t} e^{-\lambda_R t}$). Thus $U_t = Y_t$.

(ii). There is no jump of $N_{R,t}$ in $(0, t]$ and there is exactly one jump of $N_{P,t}$ in $(0, t]$ (the probability that this case occurs is $e^{-\lambda_R t} \lambda_P t e^{-\lambda_P t}$), with claim amount z , and

(a) $z < Y_t$, i.e. ruin does not occur and, thus $U_t = Y_t - z$.

(b) $z > Y_t$, i.e. ruin occurs due to the claim, or

(c) $z = Y_t$, i.e. ruin occurs due to oscillation (the probability that this case occurs is zero).

(iii) There is no jump of $N_{P,t}$ in $(0, t]$ and there is exactly one jump of $N_{R,t}$ in $(0, t]$ (the probability that this case occurs is $e^{-\lambda_P t} \lambda_R t e^{-\lambda_R t}$), and thus $U_t = (1 + S_{R,1})Y_t$.

(iv) Both $N_{P,t}$ and $N_{R,t}$ have one jump in $(0, t]$ (the probability that this case occurs is $o(t)$).

(v) $N_{P,t}$ and/or $N_{R,t}$ has more than one jumps in $(0, t]$ (the probability that this case occurs is $o(t)$).

By considering the five possible cases above and noticing that in case (ii)(b), $\phi(Y_t - z) = w(Y_t, z - Y_t)$, we have

$$\begin{aligned}
\phi(u) &= e^{-\delta t} e^{-\lambda_P t} e^{-\lambda_R t} \mathbb{E}_u \phi(Y_t) \\
&\quad + e^{-\delta t} (\lambda_P t e^{-\lambda_P t}) e^{-\lambda_R t} \mathbb{E}_u \int_{-\infty}^{Y_t} \phi(Y_t - z) dF_P(z) \\
&\quad + e^{-\delta t} (\lambda_P t e^{-\lambda_P t}) e^{-\lambda_R t} \mathbb{E}_u \int_{Y_t}^{\infty} w(Y_t, z - Y_t) dF_P(z) \\
&\quad + e^{-\delta t} e^{-\lambda_P t} (\lambda_R t e^{-\lambda_R t}) \mathbb{E}_u \int_{-1}^{\infty} \phi(Y_t(1 + z)) dF_R(z) + o(t). \tag{3.12}
\end{aligned}$$

Note that $e^{-\delta t}e^{-\lambda_P t}e^{-\lambda_R t} = 1 - (\delta + \lambda_P + \lambda_R)t + o(t)$, we have

$$\begin{aligned}
0 &= \mathbb{E}_u \phi(Y_t) - \phi(u) - (\delta + \lambda_P + \lambda_R)t \mathbb{E}_u \phi(Y_t) \\
&\quad + e^{-\delta t}(\lambda_P t e^{-\lambda_P t})e^{-\lambda_R t} \mathbb{E}_u \int_{-\infty}^{Y_t} \phi(Y_t - z) dF_P(z) \\
&\quad + e^{-\delta t}(\lambda_P t e^{-\lambda_P t})e^{-\lambda_R t} \mathbb{E}_u \int_{Y_t}^{\infty} w(Y_t, z - Y_t) dF_P(z) \\
&\quad + e^{-\delta t}e^{-\lambda_P t}(\lambda_R t e^{-\lambda_R t}) \mathbb{E}_u \int_{-1}^{\infty} \phi(Y_t(1+z)) dF_R(z) + o(t). \tag{3.13}
\end{aligned}$$

By Itô's formula, we have

$$\lim_{t \rightarrow 0} \frac{\mathbb{E}_u \phi(Y_t) - \phi(u)}{t} = \frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2)\phi''(u) + (p + ru)\phi'(u). \tag{3.14}$$

Therefore, by dividing t on both sides of (3.13), letting $t \rightarrow 0$, and using (3.14), we get (3.5). Let

$$Z_t = \int_0^t \mathcal{E}_1(R)_{s-}^{-1} dP_s - \rho\sigma_P\sigma_R \int_0^t \mathcal{E}_1(R)_{s-}^{-1} ds,$$

and $Z_\infty = \lim_{t \rightarrow \infty} Z_t$. Since $p - \rho\sigma_P\sigma_R - \lambda_P \mathbb{E}[S_{P,i}] > 0$, it follows from Theorem 3.1 in Paulsen (1993) that Z_t is a submartingale. In addition, the conditions $\sigma_P^2 > 0$, $F_R(-1) = 0$ and $r - \frac{1}{2}\sigma_R^2 > 0$ imply that Z_∞ exists and is finite with probability one. Thus, from (2.4) we find that U_t drift to $+\infty$ with probability one. Consequently, $\psi(+\infty) = 0$. The boundary condition $\lim_{u \rightarrow \infty} \phi(u) = 0$ follows from $\phi(u) \leq M\psi(u)$, where M is a upper bound of $w(x_1, x_2)$; The boundary condition $\phi(0+) = w(0, 0)$ follows from the oscillating nature of the sample paths of U_t . The results (ii) and (iii) can be proved by the same arguments as (i).

Remark 3.1. *Let us compare our results with known results.*

- (1). Letting $\rho = 0, \delta = 0, F_P(0) = 0$ and $w(x, y) \equiv 1$ in (3.5) we get the result Theorem 2.1 (i) in Paulsen and Gjessing (1997a); Letting $\rho = 0, F_P(0) = 0$ and $w(x, y) \equiv 1$ in (3.5) we get the result Theorem 2.1 (ii) in Paulsen and Gjessing (1997a).
- (2). Letting $\rho = 0, \sigma_R = 0, \lambda_R = 0, F_P(0) = 0$ and $w(x, y) \equiv 1$ in (3.5), (3.7) and (3.9), we get the result (3.1), (3.4) and (3.9) in Cai and Yang (2005), respectively.

Example 3.1. Under the assumptions of Theorem 3.1, assume that $\delta = \lambda_P = \lambda_R = 0$ and $w(x, y) \equiv 1$, then for any $u > 0$, $\psi(u)$ and $\psi_d(u)$ satisfy the same differential equation

$$\frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2)h''(u) + (p + ru)h'(u) = 0$$

and the following boundary conditions

$$h(0) = 1, \quad h(+\infty) = 0,$$

where $h(u) = \psi(u)$ or $\psi_d(u)$.

If $|\rho| < 1$, the solution was found by Hailiang Yang; See Cai and Xu (2006, p.130). If $|\rho| = 1$, the solution is given by $h(u) = 1 - \frac{K(u)}{K(\infty)}$, where

$$K(u) = \int_0^u \left(v + \frac{\rho\sigma_P}{\sigma_R} \right)^{-\frac{2r}{\sigma_R^2}} \exp \left\{ \left(\frac{2r\rho\sigma_P}{\sigma_R} - p \right) \left(v + \frac{\rho\sigma_P}{\sigma_R} \right)^{-1} \right\} dv.$$

Example 3.2. Under the assumptions of Theorem 3.1, assume that $\lambda_p = \lambda_R = 0$, $w(x, y) \equiv 1$ and $\delta > r$, then for any $u > 0$, $\phi(u)$ and $\phi_d(u)$ satisfy the same differential equation

$$\frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2)g''(u) + (p + ru)g'(u) = \delta g(u) \quad (3.15)$$

and the following boundary conditions

$$g(0) = 1, \quad g(+\infty) = 0,$$

where $g(u) = \phi(u)$ or $\phi_d(u)$.

A change of variables $x = u + \frac{\sigma_P}{\sigma_R}\rho$ and $h(x) = g(u)$ brings the equation (3.15) into the form

$$\frac{1}{2} [\sigma_P^2(1 - \rho^2) + \sigma_R^2 x^2] h''(x) + \kappa(x)h'(x) = \delta h(x), \quad (3.16)$$

where

$$\kappa(x) = p - \frac{r\rho\sigma_P}{\sigma_R} + rx.$$

When $\rho^2 < 1$, (3.16) has the same form as (A1) in Paulsen and Gjessing (1997a), using Theorem A.1. in Paulsen and Gjessing (1997a) we get

$$h(x) = C_1 D(x, \alpha + 1) + C_2 E(x, \alpha + 1),$$

where

$$D(x, \lambda) = \frac{\int_{\arctan((\sigma_R/\sigma_P)x)}^{\frac{\pi}{2}} (\cos t)^{\beta-\lambda} (\sigma_P \sin t - \sigma_R x \cos t)^\lambda \exp \left\{ -\frac{2p}{\sigma_P \sigma_R} t \right\} dt}{\sigma_P^{1+\beta} (\sigma_R)^{1+\lambda}},$$

$$E(x, \lambda) = \frac{\int_{-\frac{\pi}{2}}^{\arctan((\sigma_R/\sigma_P)x)} (\cos t)^{\beta-\lambda} (\sigma_R x \cos t - \sigma_P \sin t)^\lambda \exp \left\{ -\frac{2p}{\sigma_P \sigma_R} t \right\} dt}{\sigma_P^{1+\beta} (\sigma_R)^{1+\lambda}}.$$

Here

$$\beta = \sqrt{\left(\frac{2r}{\sigma_R^2} - 1 \right)^2 + 8\frac{\delta}{\sigma_R^2}} - 1 \quad (\text{Re}(\beta) > 0),$$

$$\alpha = \frac{1}{2} \left\{ \sqrt{\left(\frac{2r}{\sigma_R^2} - 1\right)^2 + 8\frac{\delta}{\sigma_R^2}} - \left(1 + \frac{2r}{\sigma_R^2}\right) \right\} \quad (\text{Re}(\alpha) > 0).$$

Because $E(x, \alpha + 1) \rightarrow +\infty$ as $x \rightarrow +\infty$ and $h(+\infty) = g(+\infty) = 0$, so that $C_2 = 0$. In addition, using boundary condition $g(0) = 1$, we get

$$C_1 = \frac{1}{D\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right)}.$$

Thus,

$$g(u) = h(x) = \frac{D\left(u + \frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right)}{D\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right)}.$$

This result is also obtained by Paulsen and Gjessing (1997a) in the case where $\rho = 0$.

4 Total discounted dividends

4.1 Threshold strategy

In this subsection, we consider the threshold strategy for dividend payments. More specifically we assume that the company pays dividends according to the following strategy governed by parameters $b > 0$ and $\mu > 0$. Whenever the modified surplus is below the threshold level b , no dividends are paid. However, when the surplus is above this threshold level, dividends are paid at a constant rate μ . Once the surplus is negative, the company is ruined and the process stops. We assume that the risk process U without dividends follows (2.3). We define the modified risk process $U_b = \{U_b(t) : t \geq 0\}$ in which $U_b(t)$ is the solution of stochastic differential equation

$$dU_b(t) = dP_t + U_b(t-)\mathrm{d}R_t - \mathrm{d}D_b(t), \quad t \geq 0.$$

where $D_b(t) = \mu \int_0^t I(U_b(s) > b) \mathrm{d}s$. Let $D_1(b)$ denote the present value of all dividends until time of ruin T_1 ,

$$D_1(b) = \int_0^{T_1} e^{-\delta t} \mathrm{d}D_b(t),$$

where $T_1 = \inf\{t > 0 : U_b(t) < 0\}$ with $T_1 = \infty$ if $U_b(t) \geq 0$ for all $t \geq 0$. Here $\delta > 0$ is the discount factor. Denote by $V(u; b)$ the expected discounted value of dividend payments, that is,

$$V(u) := V(u; b) = \mathbb{E}[D_1(b) | U_b(0) = u] \equiv \mathbb{E}_u[D_1(b)].$$

Let

$$M_1(u, y; b) = \mathbb{E}[e^{yD_1(b)} | U_b(0) = u] \equiv \mathbb{E}_u[e^{yD_1(b)}]$$

denote the moment-generating function of $D_1(b)$. If $\delta > 0$, then $0 \leq D_1(b) \leq \frac{\mu}{\delta}$, and thus $M(u, y; b)$ exists for all finite y .

Theorem 4.1. *Assume that $V(u)$ is twice continuously differentiable on $(0, b) \cup (b, \infty)$. Then for $0 < u < b$, $V(u)$ satisfies the following integro-differential equation:*

$$(\delta + \lambda_P + \lambda_R)V(u) = \mathcal{G}V(u), \quad (4.1)$$

and for $u > b$, $V(u)$ satisfies the following integro-differential equation:

$$(\delta + \lambda_P + \lambda_R)V(u) = \mathcal{G}V(u) - \mu V'(u) + \mu, \quad (4.2)$$

where \mathcal{G} is defined by (3.4).

Proof. Let $V_m(u)$ be twice continuously differentiable and equals to $V(u)$ on $(-\infty, b - \frac{1}{m}] \cup [b + \frac{1}{m}, \infty)$. Applying Itô's formula for semimartingales to deduce that for $t \in [0, T_1)$

$$e^{-\delta t} V_m(U_b(t)) = V_m(U_b(0)) + \int_0^t e^{-\delta s} (\mathcal{L}_\mu - \delta) V_m(U_b(s)) ds + M_t^m,$$

where M_t^m is a local martingale and \mathcal{L}_μ is defined as $\mathcal{L}_\mu g(y) = -\mu I(y > b)g'(y) + \mathcal{L}g(y)$, where \mathcal{L} is defined by (2.6). It follows that for any appropriate localization sequence of stopping times $\{\tau_n, n \geq 1\}$ we have

$$\mathbb{E}_u[e^{-\delta(t \wedge T_1 \wedge \tau_n)} V_m(U_b(t \wedge T_1 \wedge \tau_n))] = V_m(u; b) + \mathbb{E}_u \left[\int_0^{t \wedge T_1 \wedge \tau_n} e^{-\delta s} (\mathcal{L}_\mu - \delta) V_m(U_b(s)) ds \right]. \quad (4.3)$$

Letting $n, m \uparrow \infty$ and $t \uparrow \infty$ in (4.4) and note that $V(U_b(T_1)) = 0$, we find that

$$V(u) = \mathbb{E}_u \left[\int_0^{T_1} \mu e^{-\delta s} I(U_b(s) > b) ds \right]$$

if and only if $\mathcal{L}_\mu V(u) - \delta V(u) = -\mu I(u > b)$. From which we get (4.1) and (4.2). This ends the proof of Theorem 4.1.

Remark 4.1. *It can be verified that $V(u) = 0$ on $u < 0$, $V(0) = 0$ if $\sigma_P > 0$ and $\lim_{u \rightarrow \infty} V(u) = \frac{\mu}{\delta}$; V satisfy the continuity condition $V(b-) = V(b+) = V(b)$. Moreover, if $\sigma_P = \sigma_R = 0$ then $pV'(b-) = (p - \mu)V'(b+) + \mu$, and if $\sigma_P > 0$, then $V'(b-) = V'(b+)$.*

Remark 4.2. *The above result was obtained by Gerber and Shiu (2006) for the compound Poisson model, Wan (2007) for the compound Poisson model perturbed by diffusion and Ng (2009) for the dual of the compound Poisson model.*

Example 4.1. Assume that $\lambda_P = \lambda_R = 0$. Then $V(u)$ solves the following different equations

$$\begin{aligned} \frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2)V''(u) + (p + ru)V'(u) &= \delta V(u), \quad 0 < u < b, \\ \frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2)V''(u) + (p - \mu + ru)V'(u) + \mu &= \delta V(u), \quad u > b, \end{aligned}$$

with the boundary conditions

$$V(0) = 0, \quad \lim_{u \rightarrow \infty} V(u) = \frac{\mu}{\delta}, \quad V(b-) = V(b+), \quad V'(b-) = V'(b+).$$

Similar to Example 3.2, the solution is given by

$$V(u) = \begin{cases} C_3 D(u + \frac{\sigma_P}{\sigma_R} \rho, \alpha + 1) + C_4 E(u + \frac{\sigma_P}{\sigma_R} \rho, \alpha + 1), & \text{if } u \leq b, \\ C_5 D_1(u + \frac{\sigma_P}{\sigma_R} \rho, \alpha + 1) + C_6 E_1(u + \frac{\sigma_P}{\sigma_R} \rho, \alpha + 1) + \frac{\mu}{\delta}, & \text{if } u > b, \end{cases}$$

where D, E, α and β are defined in Example 3.2 and

$$\begin{aligned} D_1(x, \lambda) &= \frac{\int_{\arctan((\sigma_R/\sigma_P)x)}^{\frac{\pi}{2}} (\cos t)^{\beta-\lambda} (\sigma_P \sin t - \sigma_R x \cos t)^\lambda \exp \left\{ -\frac{2(p-\mu)}{\sigma_P \sigma_R} t \right\} dt}{\sigma_P^{1+\beta} (\sigma_R)^{1+\lambda}}, \\ E_1(x, \lambda) &= \frac{\int_{-\frac{\pi}{2}}^{\arctan((\sigma_R/\sigma_P)x)} (\cos t)^{\beta-\lambda} (\sigma_R x \cos t - \sigma_P \sin t)^\lambda \exp \left\{ -\frac{2(p-\mu)}{\sigma_P \sigma_R} t \right\} dt}{\sigma_P^{1+\beta} (\sigma_R)^{1+\lambda}}. \end{aligned}$$

The constants $C_3 - C_6$ can be determined by the boundary conditions above and they are given by $C_6 = 0$,

$$\begin{aligned} C_3 &= \frac{\frac{\mu}{\delta} D_1(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha) E(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1)}{Q_1 E(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) - Q_2 D(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1)}, \\ C_4 &= -\frac{\frac{\mu}{\delta} D(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) D_1(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha)}{Q_1 E(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) - Q_2 D(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1)}, \\ C_5 &= \frac{\frac{\mu}{\delta} \left(E(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) D(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha) + D(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) E(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha) \right)}{Q_1 E(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) - Q_2 D(\frac{\rho\sigma_P}{\sigma_R}, \alpha + 1)}. \end{aligned}$$

Here

$$\begin{aligned} Q_1 &= D(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) D_1(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha) - D(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha) D_1(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha + 1), \\ Q_2 &= D_1(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha) E(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) + D_1(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha + 1) E(b + \frac{\rho\sigma_P}{\sigma_R}, \alpha). \end{aligned}$$

Theorem 4.2. Assume that $M_1(u, y; b)$ is twice continuously differentiable in u on $(0, b) \cup (b, \infty)$ and once in $y \geq 0$. Then M_1 satisfies the following integro-differential equations

$$\mathcal{A}M_1(u, y; b) - \delta y \frac{\partial M_1(u, y; b)}{\partial y} - (\lambda_R + \lambda_P) M_1(u, y; b) + \lambda_P (1 - F_P(u)) = 0, \quad 0 < u < b, \quad (4.4)$$

and

$$\begin{aligned}\mathcal{A}M_1(u, y; b) &= \mu \frac{\partial M_1(u, y; b)}{\partial u} - \delta y \frac{\partial M_1(u, y; b)}{\partial y} + \mu y M_1(u, y; b) \\ &- (\lambda_R + \lambda_P) M_1(u, y; b) + \lambda_P (1 - F_P(u)) = 0, \quad u > b,\end{aligned}\quad (4.5)$$

where

$$\begin{aligned}\mathcal{A}M(u, y; b) &= \frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2) \frac{\partial^2 M(u, y; b)}{\partial u^2} + (p + ru) \frac{\partial M(u, y; b)}{\partial u} \\ &+ \lambda_P \int_{-\infty}^u M(u - z, y; b) dF_P(z) \\ &+ \lambda_R \int_{-1}^{\infty} M(u + uz, y; b) dF_R(z).\end{aligned}\quad (4.6)$$

In addition, $M_1(u, y; b)$ satisfies

$$M_1(0, y; b) = 1, \quad (4.7)$$

$$\lim_{u \rightarrow \infty} M_1(u, y; b) = e^{y\mu/\delta}. \quad (4.8)$$

Proof. When $0 < u < b$, consider the infinitesimal time interval from 0 to t . By the Markov property of the process U_t , we have

$$\begin{aligned}M_1(u, y; b) &= E_u[e^{y \int_0^T e^{-\delta s} dD(s)}] + o(t) \\ &= E_u[e^{y \int_0^{T-t} e^{-\delta(t+s)} dD(t+s)}] + o(t) \\ &= E_u[e^{y \int_0^T e^{-\delta(t+s)} dD(s)} \circ \theta_t] + o(t) \\ &= E_u(E_{U_t}[e^{ye^{-\delta t} \int_0^T e^{-\delta s} dD(s)}]) + o(t) \\ &= E_u[M_1(U_t, ye^{-\delta t}; b)] + o(t),\end{aligned}\quad (4.9)$$

where θ_t is the shift operator. We refer to Kallenberg (2006) for more details on the Markov property and the shift operator. By the law of double expectation, we have

$$\begin{aligned}E_u[M_1(U_t, ye^{-\delta t}; b)] &= (1 - \lambda_P t)(1 - \lambda_R t) E_u[M_1(Y_t^1, ye^{-\delta t}; b)] \\ &+ \lambda_P t(1 - \lambda_R t) E_u \int_{-\infty}^{Y_t^1} M_1(Y_t^1 - z, ye^{-\delta t}; b) dF_P(z) \\ &+ \lambda_P t(1 - \lambda_R t) E_u(1 - F_P(Y_t^1)) \\ &+ \lambda_R t(1 - \lambda_P t) E_u \int_{-1}^{\infty} M_1(Y_t^1(1 + z), ye^{-\delta t}; b) dF_R(z) \\ &+ o(t),\end{aligned}\quad (4.10)$$

where

$$Y_t^1 = ue^{\Delta t} + pe^{\Delta t} \int_0^t e^{-\Delta s} ds + \sigma_P e^{\Delta t} \int_0^t e^{-\Delta s} dW_{P,s}.$$

Here

$$\Delta_t = \left(r - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t}.$$

By Itô's formula and note that

$$\begin{aligned} Y_t^1 &\stackrel{d}{=} u + \int_0^t (p + rY_s)ds \\ &+ \int_0^t \sqrt{(\sigma_P + \rho\sigma_R Y_{s-}^1)^2 + \sigma_R^2(1 - \rho^2)(Y_{s-}^1)^2} dB_s, \end{aligned}$$

we have

$$\begin{aligned} dM_1(Y_t^1, e^{-\delta t}y; b) &= \frac{\partial M_1(Y_t^1, e^{-\delta t}y; b)}{\partial u} dY_t^1 + \frac{\partial M_1(Y_t^1, e^{-\delta t}y; b)}{\partial y} y de^{-\delta t} \\ &+ \frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R Y_{t-}^1 + \sigma_R^2(Y_{t-}^1)^2) \frac{\partial^2 M_1(Y_t^1, e^{-\delta t}y; b)}{\partial u^2}. \end{aligned}$$

Thus

$$\begin{aligned} E_u M_1(Y_t^1, e^{-\delta t}y; b) &= M_1(u, y; b) + E_u \int_0^t (p + rY_s) \frac{\partial M_1(Y_s^1, e^{-\delta s}y; b)}{\partial u} ds \\ &+ \frac{1}{2} E_u \int_0^t \frac{\partial^2 M_1(Y_s^1, e^{-\delta s}y; b)}{\partial u^2} (\sigma_P^2 + 2\rho\sigma_P\sigma_R Y_{s-}^1 + \sigma_R^2(Y_{s-}^1)^2) ds \\ &- \delta y E_u \int_0^t e^{-\delta s} \frac{\partial M_1(Y_s^1, e^{-\delta s}y; b)}{\partial y} ds. \end{aligned} \quad (4.11)$$

Substituting (4.11) into (4.10) and then dividing both sides of (4.10) by t , letting $t \rightarrow 0$ and rearranging, we obtain (4.4).

Similarly, when $u > b$ we have

$$M_1(u, y; b) = e^{y\mu t} E_u [M_1(U_t, ye^{-\delta t}; b)] + o(t), \quad (4.12)$$

from which we obtain

$$\begin{aligned} M_1(u, y; b) &= (1 - \lambda_P t)(1 - \lambda_R t) e^{y\mu t} E_u [M_1(Y_t^2, ye^{-\delta t}; b)] \\ &+ \lambda_P t(1 - \lambda_R t) e^{y\mu t} E_u \int_{-\infty}^{Y_t^2} M_1(Y_t^2 - z, ye^{-\delta t}; b) dF_P(z) \\ &+ \lambda_P t(1 - \lambda_R t) e^{y\mu t} E_u (1 - F_P(Y_t^2)) \\ &+ \lambda_R t(1 - \lambda_P t) e^{y\mu t} E_u \int_{-1}^{\infty} M_1(Y_t^2(1 + z), ye^{-\delta t}; b) dF_R(z) \\ &+ o(t), \end{aligned} \quad (4.13)$$

where

$$Y_t^2 = ue^{\Delta_t} + (p - \mu)e^{\Delta_t} \int_0^t e^{-\Delta_s} ds + \sigma_P e^{\Delta_t} \int_0^t e^{-\Delta_s} dW_{P,s},$$

$$\Delta_t = \left(r - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t}.$$

Using the same argument as for (4.4) we get (4.5). The condition (4.7) is obvious, and (4.8) follows from $\lim_{u \rightarrow \infty} D_1(b) = \frac{\mu}{\delta}$. This ends the proof of Theorem 4.2.

Set

$$M_1(u, y; b) = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!} V_k(u; b), \quad (4.14)$$

where

$$V_k(u) \equiv V_k(u; b) = E[D_1(b)^k | U_b(0) = u], \quad (4.15)$$

is the k th moment of $D_1(b)$. Substitution of (4.14) into (4.4) and (4.5) and comparing the coefficients of y^k yields the following integro-differential equations

$$\mathcal{G}V_k(u) = (\lambda_R + \lambda_p + k\delta)V_k(u), \quad 0 < u < b, \quad (4.16)$$

and

$$\mathcal{G}V_k(u) - \mu \frac{\partial V_k(u)}{\partial u} + k\mu V_{k-1}(u) = (\lambda_R + \lambda_p + k\delta)V_k(u), \quad u > b, \quad (4.17)$$

where \mathcal{G} is defined by (3.4). They generalize (4.1) and (4.2), which are for $k = 1$. The boundary conditions are $V_k(0; b) = 0$ and $\lim_{u \rightarrow \infty} V_k(u; b) = (\frac{\mu}{\delta})^k$.

4.2 Barrier strategy

It is assumed that dividends are paid according to a barrier strategy ξ_b . Such a strategy has a level of the barrier $b > 0$, when the surplus exceeds the barrier, the excess is paid out immediately as the dividend. When the surplus is below b , nothing is done. Let D_t^b be aggregated dividends up to time t by insurance company whose risk process is modeled by (2.3). The controlled risk process when taking into account of the dividend strategy ξ_b is $U^b = \{U_t^b : t \geq 0\}$, where U_t^b is the solution of stochastic differential equation

$$dU_t^b = dP_t + U_{t-}^b dR_t - dD_t^b, \quad t \geq 0.$$

Denote by $\bar{V}_1(u; b)$ the dividend-value function if barrier strategy ξ_b is applied, that is,

$$\bar{V}_1(u; b) = E_u[D_2(b)],$$

where $D_2(b) = \int_0^{T_2} e^{-\delta t} dD_t^b$. Here $\delta > 0$ is the force of interest for valuation and $T_2 = \inf\{t \geq 0 : U_t^b < 0\}$. Let

$$M_2(u, y; b) = E_u[e^{yD_2(b)}]$$

denote the moment-generating function of $D_2(b)$. $M(u, y, b)$ exists for all finite y .

Theorem 4.3. Assume that $M_2(u, y; b)$ is twice continuously differentiable in u on $(0, b)$ and once in $y \geq 0$. Then M_2 satisfies the following integro-differential equation

$$\mathcal{A}M_2(u, y; b) - \delta y \frac{\partial M_2(u, y; b)}{\partial y} - (\lambda_R + \lambda_P)M_2(u, y; b) + \lambda_P(1 - F_P(u)) = 0, \quad 0 < u < b, \quad (4.18)$$

where \mathcal{A} is defined by (4.6). In addition, $M_2(u, y; b)$ satisfies

$$M_2(0, y; b) = 1, \quad (4.19)$$

$$\frac{\partial M_2(u, y; b)}{\partial u} \Big|_{u=b} = yM_2(b, y; b). \quad (4.20)$$

Proof. The proof of (4.18) is same as the proof of (4.4). The condition (4.19) is obvious. To prove (4.20), we first consider the special case in which $\sigma_P = \sigma_R = 0$. Letting $u \uparrow b$ in (4.18) gives

$$\begin{aligned} & (p + rb) \frac{\partial M_2(u, y; b)}{\partial u} \Big|_{u=b} - \delta y \frac{\partial M_2(u, y; b)}{\partial y} \Big|_{u=b} - (\lambda_R + \lambda_P)M_2(b, y; b) \\ & + \lambda_P \int_{-\infty}^b M_2(b - z, y; b) dF_P(z) + \lambda_P(1 - F_P(b)) \\ & + \lambda_R \int_{-1}^{\infty} M_2(b(1 + z), y; b) dF_R(z) = 0. \end{aligned} \quad (4.21)$$

Similarly, for $u = b$ we have

$$\begin{aligned} M_2(b, y; b) &= e^{-\lambda_P t} e^{-\lambda_R t} e^{y(p+rb)t} M_2(b, ye^{-\delta t}; b) \\ &+ \lambda_P t e^{-\lambda_P t} e^{-\lambda_R t} e^{y(p+rb)t} \int_{-\infty}^b M_2(b - z, ye^{-\delta t}; b) dF_P(z) \\ &+ \lambda_P t e^{-\lambda_P t} e^{-\lambda_R t} e^{y(p+rb)t} (1 - F_P(b)) \\ &+ e^{-\lambda_P t} \lambda_R t e^{-\lambda_R t} e^{y(p+rb)t} \int_{-1}^{\infty} M_2(b(1 + z), ye^{-\delta t}; b) dF_R(z) \\ &+ o(t). \end{aligned} \quad (4.22)$$

This, together with the following Taylor's expansion

$$M_2(b, ye^{-\delta t}; b) = M_2(b, y; b) - \delta y t \frac{\partial M_2(b, y; b)}{\partial y} + o(t).$$

gives

$$\begin{aligned} -\delta y \frac{\partial M_2(u, y; b)}{\partial y} \Big|_{u=b} &- (\lambda_R + \lambda_P - y(p + rb))M_2(b, y; b) \\ &+ \lambda_P \int_{-\infty}^b M_2(b - z, y; b) dF_P(z) + \lambda_P(1 - F_P(b)) \\ &+ \lambda_R \int_{-1}^{\infty} M_2(b(1 + z), y; b) dF_R(z) = 0. \end{aligned} \quad (4.23)$$

Comparing (4.23) with (4.21) we obtain

$$\frac{\partial M_2(u, y; b)}{\partial u} \Big|_{u=b} = yM_2(b, y; b).$$

If $\sigma_P^2 > 0, \sigma_R^2 = 0$ or $\sigma_P^2 > 0, \sigma_R^2 > 0$, the process can be viewed as the limit of a family of same kind processes without Brownian motions, and in this way (4.20) can be obtained as limiting results. This ends the proof of Theorem 4.3.

Set

$$M_1(u, y; b) = 1 + \sum_{k=1}^{\infty} \frac{y^k}{k!} \bar{V}_k(u; b), \quad (4.24)$$

where

$$\bar{V}_k(u) \equiv \bar{V}_k(u; b) = \mathbb{E}_u[D_2(b)^k],$$

is the k th moment of $D_2(b)$. Substitution of (4.24) into (4.18) and comparing the coefficients of y^k yields the following integro-differential equation

$$\mathcal{G}\bar{V}_k(u) = (\lambda_R + \lambda_p + k\delta)\bar{V}_k(u), \quad 0 < u < b,$$

where \mathcal{G} is defined by (3.4). The boundary conditions are $V_k(0; b) = 0$ and

$$\frac{\partial \bar{V}_k(u; b)}{\partial u} \Big|_{u=b} = k\bar{V}_{k-1}(b; b).$$

Example 4.2. Assume that $\lambda_P = \lambda_R = 0$. If $u \leq b$, then $\bar{V}_1(u; b)$ solves the following differential equation

$$\frac{1}{2}(\sigma_P^2 + 2\rho\sigma_P\sigma_R u + \sigma_R^2 u^2) \frac{\partial^2 \bar{V}_1(u; b)}{\partial u^2} + (p + ru) \frac{\partial \bar{V}_1(u; b)}{\partial u} = \delta \bar{V}_1(u; b) \quad (4.25)$$

with

$$\bar{V}_1(0; b) = 0, \quad \frac{\partial \bar{V}_1(u; b)}{\partial u} \Big|_{u=b} = 1. \quad (4.26)$$

When $\rho^2 < 1$, the solution of (4.25) is given by

$$\bar{V}_1(u; b) = C_7 D\left(u + \frac{\sigma_P}{\sigma_R} \rho, \alpha + 1\right) + C_8 E\left(u + \frac{\sigma_P}{\sigma_R} \rho, \alpha + 1\right),$$

where D, E and α are defined in Example 3.2. The constants C_7 and C_8 can be determined by conditions (4.26). Using that $\frac{\partial}{\partial y} D(y, \alpha) = -\alpha D(y, \alpha - 1)$ and $\frac{\partial}{\partial y} E(y, \alpha) = \alpha E(y, \alpha - 1)$ we obtain

$$C_7 = -\frac{E\left(\frac{\sigma_P}{\sigma_R} \rho, \alpha + 1\right)}{(\alpha + 1)A\left(b + \frac{\sigma_P}{\sigma_R} \rho, \alpha\right)},$$

$$C_8 = \frac{D\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right)}{(\alpha + 1)A\left(b + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right)},$$

where

$$\begin{aligned} A\left(b + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right) = & E\left(b + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right) D\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right) \\ & + D\left(b + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right) E\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right). \end{aligned}$$

It follows that

$$\bar{V}_1(u; b) = \frac{B\left(u + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right)}{(\alpha + 1)A\left(b + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right)},$$

where

$$\begin{aligned} B\left(u + \frac{\sigma_P}{\sigma_R}\rho, \alpha\right) = & D\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right) E\left(u + \frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right) \\ & - E\left(\frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right) D\left(u + \frac{\sigma_P}{\sigma_R}\rho, \alpha + 1\right). \end{aligned}$$

In particular, when $\rho = 0$ we recover the result of Example 2.2 in Paulsen and Gjessing (1997b).

5 Concluding remarks

In this paper, a generalized Paulsen-Gjessing's risk model is examined, some rather general integro-differential equations satisfied by the Gerber-Shiu functions, the expected discounted dividends up to ruin and the moment generating functions of the discounted dividends are presented, respectively. Generally speaking, it is difficult to find the analytical solutions except for some specials. A numerical method called the block-by-block has been used by Paulsen et al. (2005) to find the probability of ultimate ruin in the classical risk model with stochastic return on investments. The solutions, either analytical or numerical, of the integro-differential equations in this paper are not only interesting but also valuable in practice. Other research problems such as the optimality results for dividend and investment need also to be studied.

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